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Three new upper bounds on the chromatic number

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ABSTRACT

This paper introduces three new upper bounds on the chromatic number, without making any assumptions on the graph structure. The first one, ξ, is based on the number of edges and nodes, and is to be applied to any connected component of the graph, whereas ζ and η are based on the degree of the nodes in the graph. The computation complexity of the three-bound computation is assessed. Theoretical and computational comparisons are also made with five well-known bounds from the literature, which demonstrate the superiority of the new upper bounds.

1. Introduction

An undirected, simple graph is given; the graph coloring problem is to assign a color to every node in such a way that two adjacent nodes do not have the same color, while minimizing the total number of colors used. This problem arises in many practical applications, such as map coloring, timetabling, scheduling, memory allocation, and many others [6]. Formally, a coloring of graph $G = (X, U)$ is a function $F : X \to \mathbb{N}^*$, where each node in X is allocated an integer value that is called a color. A proper coloring satisfies $F(u) \neq F(v)$ for all $(u, v) \in U$ [4,7]. A graph is said to be α-colorable if there exists a coloring which uses, at most, α different colors. In that case, all the nodes colored with the same color are said to be part of the same class. The smallest number of colors involved in any proper coloring of a graph G is called the *chromatic number*, which is denoted by $\chi(G)$. The problem of finding $\chi(G)$, as well as a minimum coloring, is $\mathcal{NP}$-hard and is still the focus of an intense research effort [2,3,9,10].

First, we recall some elementary results on the graph coloring problem, and introduce some notations. A graph cannot be α-colorable with $\alpha < \chi(G)$. The chromatic number equals 1, if and only if G is a totally disconnected graph. It is equal to $|X|$ if G is complete, and for the graphs that are exactly bipartite (including trees and forests), the chromatic number is 2.

Let G be a non directed, simple graph, where $n = |X|$ is the number of nodes, and $m = |U|$ is the number of edges. The degree of node i is denoted by d_i for all $i \in \{1, \ldots, n\}$, and $\delta(G)$ is the highest degree in G. The following upper bounds on $\chi(G)$ can be found in the literature:

- $\chi(G) \leq d = \delta(G) + 1$ [4,6].
- $\chi(G) \leq l = \left\lfloor \frac{1+\sqrt{8m+1}}{2} \right\rfloor$ [4,6].
- $\chi(G) \leq M = \max_{i\in X} \min(d_i + 1, i)$, provided that $d_1 \geq d_2 \geq \cdots \geq d_n$ [13].
- $\chi(G) \leq s = \delta_2(G) + 1$, where $\delta_2(G)$ is the largest degree that a node v can have, if v is adjacent to a node whose degree is at least as large as its own [11].
- $\chi(G) \leq q = \left\lceil \frac{r}{r+1}(\delta(G) + 1) \right\rceil$, where r is the maximum number of nodes of the same degree, each at least $(\delta(G) + 2)/2$ [12].

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Regarding lower bounds, the chromatic number is greater than or equal to the clique number denoted by $\omega(G)$, which is the size of the largest clique in the graph; thus $\omega(G) \leq \chi(G)$. However, this bound is difficult to use in practice as finding the clique number is $\mathcal{NP}$-hard, and the Lovasz number is known to be a better lower bound for $\chi(G)$ as it is "sandwiched" between the clique number and the chromatic number [8]. Moreover, the Lovasz number can be calculated in polynomial time.

There exist some upper bounds on the chromatic number for special classes of graphs:

- $\chi(G) \leq \delta(G)$, for a connected, simple graph which is neither complete, nor has an odd cycle.
- $\chi(G) \leq 4$, for any planar graph.

In Section 2, three new upper bounds on the chromatic number are proposed. The quality of these bounds is then compared with existing bounds in Section 3, and computational experiments are conducted in Section 4 for assessing the practical improvement of the three new upper bounds.

2. Three new upper bounds on the chromatic number

The following lemma is required for proving Theorem 1, which introduces the first bound proposed in this paper.

Lemma 1. *The following inequality holds for any connected, simple graph $G_n = (V, E)$, where $m_n = |E|$.*

$$\frac{\chi(G_n)\,(\chi(G_n)-1)}{2} + n - \chi(G_n) \leq m_n. \tag{1}$$

This inequality is referred to as Eq. (1).

Proof. Lemma 1 is proved by recurrence on n.

First, it can be observed that Lemma 1 is obviously true for $n = 2$. Indeed, there exists a unique connected, simple graph on two vertices which has a single edge, and $\chi(G_2) = 2$.

Second, we assume that Lemma 1 is valid for all graphs having at most n vertices. We now prove that the inequality Lemma 1 holds for any connected, simple graphs on $n + 1$ vertices. Let such a graph be denoted by G_{n+1}. It has m_{n+1} edges and its chromatic number is $\chi(G_{n+1})$.

G_{n+1} can be seen as a connected, simple graph G_n plus an additional vertex denoted by $n+1$, and additional edges incident to this new vertex. The addition of vertex $n + 1$ to G_n either leads to $\chi(G_{n+1}) = \chi(G_n)$, or to $\chi(G_{n+1}) = \chi(G_n) + 1$. Indeed, the introduction of a new vertex (along with its incident edges) into a graph leads to increment of the chromatic number by at most one.

- First case: $\chi(G_{n+1}) = \chi(G_n)$.
  Adding 1 to Eq. (1) yields
  $$\frac{\chi(G_{n+1})\,(\chi(G_{n+1})-1)}{2} + n + 1 - \chi(G_{n+1}) \leq 1 + m_n \leq m_{n+1}.$$
  We have $1 + m_n \leq m_{n+1}$, because at least one new edge is to be added to G_n for building G_{n+1}: vertex $n + 1$ has to be connected to at least one edge in G_n for G_{n+1} to be connected.
- Second case: $\chi(G_{n+1}) = \chi(G_n) + 1$.
  A minimal coloring of G_{n+1} can be obtained by keeping the minimal coloring of G_n, and by assigning color $\chi(G_{n+1}) = \chi(G_n) + 1$ to vertex $n + 1$. Since this coloring is minimal, there exists at least one edge between any pair of color classes [4]. In particular, this requirement for color $\chi(G_{n+1})$ implies that the degree of vertex $n + 1$ is at least $\chi(G_n)$, hence $m_n + \chi(G_n) \leq m_{n+1}$.
  Adding $\chi(G_n)$ to Eq. (1) yields
  $$\left(\frac{\chi(G_n)\,(\chi(G_n)-1)}{2} + \chi(G_n)\right) + n - \chi(G_n) \leq m_n + \chi(G_n).$$
  The quantity in parenthesis is equal to the sum of the integers in $\{1, \ldots, \chi(G_n)\}$, and since $\chi(G_{n+1}) = \chi(G_n) + 1$,
  $$\frac{\chi(G_{n+1})\,(\chi(G_{n+1})-1)}{2} + n - \chi(G_n) \leq m_n + \chi(G_n).$$
  Finally, as $n - \chi(G_n) = n + 1 - \chi(G_{n+1})$ and $m_n + \chi(G_n) \leq m_{n+1}$,
  $$\frac{\chi(G_{n+1})\,(\chi(G_{n+1})-1)}{2} + n + 1 - \chi(G_{n+1}) \leq m_{n+1}. \quad \square$$

Theorem 1. *The following inequality holds for any connected, simple undirected graph G*

$$\chi(G) \leq \xi,$$

with $\xi = \left\lfloor \frac{3+\sqrt{9+8(m-n)}}{2} \right\rfloor$.

Proof. By Lemma 1, m can be lower bounded as follows:

$$\frac{\chi(G)(\chi(G)-1)}{2}+n-\chi(G)\leq m.$$

This inequality leads to the following second order polynomial in the variable $\chi(G)$:

$$\chi(G)^2-3\chi(G)-2(m-n)\leq 0.$$

Once solved, this inequality leads to:

$$\chi(G)\leq\left\lfloor\frac{3+\sqrt{9+8(m-n)}}{2}\right\rfloor. \quad \square$$

Note that, because all connected graphs have at least $n-1$ edges, then $8(m-n)+9\geq 1$; thus the square root is in $\mathbb{R}^+$.

Remark 1. As this bound is only based on the number of the nodes and edges in the graph, it yields the same value for all graphs having the same number of nodes and edges. This bound computation requires $\mathcal{O}(1)$ operations.

Theorem 2. *For any simple, undirected graph G, $\chi(G)\leq\zeta$, where ζ is the greatest number of nodes with a degree greater than or equal to $\zeta-1$.*

Theorem 3. *For any simple, undirected graph G, $\chi(G)\leq\eta$, where η is the greatest number of nodes with a degree greater than or equal to η that are adjacent to at least $\eta-1$ nodes, each of them with a degree larger than or equal to $\eta-1$.*

Before proving Theorems 2 and 3, some notations and definitions need to be stated. It should be noticed that connectivity is not required for the last two bounds, which involves more information on the graph topology than the first one.

The degree of saturation [1,7] of a node $v\in X$ denoted by $DS(v)$ is the number of different colors of the nodes adjacent to v. For a minimum coloring of graph G, $DS(v)$ is in $\{1,\ldots,\chi(G)-1\}$ for all $v\in X$.

The following notations are used throughout this paper.

- $C=\{1,\ldots,\chi(G)\}$ is the minimum set of colors used in any valid coloring.
- A valid (or proper) coloring using exactly $\chi(G)$ colors is said to be a minimal coloring.
- The neighborhood of node v denoted by $N(v)$ is the set of all nodes u such that edge (u,v) belongs to U. $N(v)$ is also called the set of adjacent nodes to v.

The last two bounds are based on the degree of saturation of a node and on Lemma 2.

Lemma 2. *Let F be a minimal coloring of G. For every color k in C, there exists at least one node v colored with k, (i.e., $F(v)=k$), such that its degree of saturation is $\chi(G)-1$ and where v is adjacent to at least $\chi(G)-1$ nodes with a degree larger than or equal to $\chi(G)-1$.*

Proof of Lemma 2. We prove the lemma by contradiction. First, we show that for all k in C there exists a node v, colored with k, such that $DS(v)=\chi(G)-1$. To do so, we assume that there exists a color k in C such that any node v colored with k has a degree of saturation that is strictly less than $\chi(G)-1$.

Then, it can be deduced that for all $v\in X$ such that $F(v)=k$, there exists a color $c\in C\setminus\{k\}$ such that there does not exist $u\in N(v)/F(u)=c$. Consequently, a new valid coloring can be derived from the current one by setting $F(v)=c$. Indeed, v is not connected to any node colored with c. This operation can be performed for any node colored with k, leading to a valid coloring in which color k is never used. Hence, this new coloring involves $\chi(G)-1$ colors, which is impossible by definition of the chromatic number.

Second, we show that, for every k in C, there exists a node v colored with k, whose degree of saturation is equal to $\chi(G)-1$, and such that v has at least $\chi(G)-1$ neighbors with degree larger than or equal to $\chi(G)-1$. To do so, we assume that there exists a color k in C such that any node v colored with k having a degree of saturation equal to $\chi(G)-1$ has strictly less than $\chi(G)-1$ neighbors with a degree larger than or equal to $\chi(G)-1$.

Then, it can be deduced that for all node v colored with k and such that $DS(v)=\chi(G)-1$, there exists one color $c\in C\setminus\{k\}$ such that the degree of any node $w\in V(v)/F(w)=c$ is strictly less than $\chi(G)-1$. Then, for each node $w\in V(v)/F(w)=c$, there exists a color $l\in C\setminus\{k,c\}$ such that setting $F(w)$ to l yields a valid coloring. As a result, color c is no longer used in $N(v)$, thus $DS(v)$ is no longer $\chi(G)-1$. This operation can be performed for any node v such that $F(v)=k/DS(v)=\chi(G)-1$, leading to a coloring in which there is no node v colored with k and such that $DS(v)=\chi(G)-1$. It can then be deduced from the first part of this proof that in such a situation, G can be colored with strictly less than $\chi(G)$ colors, which is impossible. $\square$

Proof of Theorem 2. It can be deduced from Lemma 2 that there exists at least $\chi(G)$ nodes in G, with a degree of at least $\chi(G)-1$. Thus, ζ being the greatest number of nodes with a degree greater than or equal to $\zeta-1$, the following inequality holds: $\chi(G)\leq\zeta$. $\square$

Remark 2. It can easily be seen that Algorithm 1, which returns ζ, has a computational complexity of $\mathcal{O}(\max\{m, n\log_2(n)\})$, as it requires enumerating the m edges to compute the degree of the nodes, $n\log_2(n)$ operations to sort the nodes, and $\zeta \leq n$ iterations in the `while` loop.

Algorithm 1: Computing ζ.

```
Data: Graph G(X, U); where n ← |X| and m ← |U|.
Compute the degree, d_i of all nodes i in X;
Sort the nodes by non increasing degree;
ζ ← 0, stable ← 0 and i ← 0;
while stable = 0 and i ≤ n do
    if d_i ≥ ζ then
        ζ ← ζ + 1;
    else
        stable ← 1;
    i ← i + 1;
```

Proof of Theorem 3. It can be deduced from Lemma 2 that, there exist at least $\chi(G)$ nodes in G, which are adjacent to $\chi(G) - 1$ nodes with degrees larger than $\chi(G) - 1$. Since η is the greatest number of nodes with a degree greater than or equal to η that are adjacent to at least $\eta - 1$ nodes, each of them with degree larger than or equal to $\eta - 1$, then $\chi(G) \leq \eta$. □

Remark 3. The proposed algorithm for computing η relies on the neighboring density. The neighboring density of node i is denoted by ρ_i and is defined as follows: ρ_i is the largest integer such that node i is adjacent to at least ρ_i nodes. Each of the latter has a degree greater than or equal to ρ_i. Algorithm 2 computes the neighboring density of all nodes. Then, η is computed by executing Algorithm 1, where d_i is replaced with ρ_i for all $i \in X$ and where ζ is replaced with η. The computational complexity for determining the neighboring density of all nodes is $\mathcal{O}(m\log_2(m))$, as it requires m operations to compute the degree, and $2m\log_2(2m)$ operations to sort $2m$ numbers (the degree sum of all nodes is $2m$). Therefore, the computational complexity for computing η is $\mathcal{O}(\max\{m\log_2(m), n\log_2(n)\})$.

Algorithm 2: Computing the neighboring density of all nodes.

```
Data: Graph G(X, U); where n ← |X| and m ← |U|.
Compute the degree of all nodes in X;
for i = 1 to n do
    Create the array tab by sorting the degree of the d_i neighbors of node i by non increasing order;
    ρ_i ← 0, stable ← 0, and j ← 0;
    while stable = 0 and j ≤ d_i do
        if tab[j] > ρ_i then
            ρ_i ← ρ_i + 1;
        else
            stable ← 1;
        j ← j + 1;
```

---

3. Theoretical quality assessment of these bounds

The three bounds introduced in this paper are compared theoretically to the five upper bounds from the literature, which were mentioned in the introduction, namely d, l, M, s and q.

Proposition 1. *For any simple, undirected, connected graph*

$$\xi \leq l.$$

Proof. The number of edges in any simple undirected graph is less than or equal to $n(n-1)/2$, thus:

$$\begin{aligned}
&2m \leq n^2 - n\\
&8m + 1 \leq 4n^2 - 4n + 1\\
&8m + 1 \leq (2n-1)^2\\
&\sqrt{8m+1} \leq 2n - 1\\
&1 - 2n \leq -\sqrt{8m+1}\\
&4 - 8n \leq -4\sqrt{8m+1}.
\end{aligned}$$

Then, $8m+5$ is added to the last inequality

$$9+8(m-n) \leq (8m+1)+4-4\sqrt{8m+1}$$
$$\sqrt{9+8(m-n)} \leq \sqrt{8m+1}-2$$
$$\frac{3+\sqrt{9+8(m-n)}}{2} \leq \frac{1+\sqrt{8m+1}}{2}$$
$$\left\lfloor \frac{3+\sqrt{9+8(m-n)}}{2} \right\rfloor \leq \left\lfloor \frac{1+\sqrt{8m+1}}{2} \right\rfloor$$
$$\xi \leq l. \quad \square$$

Proposition 2. *For any simple undirected graph*

$$\eta \leq \zeta.$$

Proof. This is obvious as the definition of ζ and η can be seen as the statement of two maximization problems. Since the requirements (or constraints) on η are more stringent than the requirements on ζ, the inequality $\eta \leq \zeta$ holds. $\square$

Proposition 3. *For any simple undirected graph*

$$\zeta \leq d.$$

Proof. Since $\delta(G)$ is the maximum degree in the graph, $d_v \leq \delta(G)$ for all $v \in X$. By definition of ζ, there exists at least one node w with a degree greater than or equal to $\zeta - 1$, then:

$$d_w \leq \delta(G)$$
$$\zeta - 1 \leq \delta(G)$$
$$\zeta \leq \delta(G) + 1$$
$$\zeta \leq d. \quad \square$$

Proposition 4. *For any simple undirected graph*

$$\zeta = M.$$

Proof. First, it is recalled that by definition of ζ, there does not exist $\zeta + 1$ nodes with a degree larger than or equal to ζ (otherwise this would be conflicting with the definition of ζ).

It is assumed without loss of generality that the nodes are indexed by non increasing degree: $d_1 \geq d_2 \geq \cdots \geq d_n$. Then it can be deduced that the nodes whose index is in $\{\zeta + 1, \ldots, n\}$ have a degree less than or equal to $\zeta - 1$.

The node set $X = \{1, \ldots, n\}$ is split into two subsets: $X = A \cup B$ with $A = \{1, \ldots, \zeta\}$ and $B = \{\zeta + 1, \ldots, n\}$. In other words, A is the set of the ζ nodes of highest degree, B is the set of the $n - \zeta$ nodes of lower degree.

For all i in X, we denote by m_i the minimum value between d_i+1 and i (i.e. this makes it possible to write $M = \max_{i \in X} m_i$).

For all $i \in X$, i is either in A or in B:

- If $i \in A$, then node i is such that $d_i \geq \zeta - 1$, i.e. $d_i + 1 \geq \zeta$. Moreover, by definition of A, $i \leq \zeta$. Consequently:

  $$m_i = i \leq \zeta \leq d_i + 1 \quad \forall i \in A.$$

  In particular, for $i = \zeta$, $m_i = \zeta$, and by definition of M, $\zeta \leq M$.
- If $i \in B$, then node i is such that $d_i \leq \zeta - 1$, i.e. $d_i + 1 \leq \zeta$. Moreover, by definition of B, $i \geq \zeta$. Consequently:

  $$m_i = d_i + 1 \leq \zeta \leq i \quad \forall i \in B.$$

Finally, the inequality $m_i \leq \zeta$ holds for all $i \in \{1, \ldots, n\}$ and by definition of M this leads to $M \leq \zeta$. $\square$

Remark 4. Computing M by using the formula $M = \max_{i \in X} \min(d_i + 1, i)$ provided in [13] has a computational complexity of $\mathcal{O}(\max\{m, n \log_2 n\})$, as it requires computing the degree of the nodes, and sorting them by non increasing degree. Although ζ and M are defined differently, their computation requires the same order of arithmetic operations.

Proposition 5. *For any simple undirected graph*

$$\eta \leq s.$$

Proof. By definition of $\delta_2(G)$, there does not exist two adjacent nodes i and j in X such that $d_i > \delta_2(G)$ and $d_j > \delta_2(G)$. Consequently, it is impossible to find a node adjacent to at least $\delta_2(G) + 1$ nodes, whose degrees are at least $\delta_2(G) + 1$. This shows that $\eta - 1$ is less than or equal to $\delta_2(G)$, i.e. $\eta \leq s$. $\square$

Table 1
Upper bounds on the chromatic number.

Instances			Known upper bounds					New upper bounds		
Sour.	Name	$n\backslash m$	d	l	M	s	q	ξ	ζ	η
MYC	myciel3	11\20	6	6	5	4	6	6	5	4
MYC	myciel4	23\71	12	12	7	7	12	11	7	6
CAR	2-Insert._3	37\72	10	12	5	5	6	10	5	5
CAR	1-FullIns_3	30\100	12	14	9	12	12	13	9	7
CAR	3-Insert._3	56\110	12	15	5	5	7	12	5	5
MIZ	mug88_1	88\146	5	17	5	5	6	12	5	4
MIZ	mug88_25	88\146	5	17	5	5	6	12	5	4
CAR	4-Insert._3	79\156	14	18	5	5	8	14	5	5
SGB	queen5_5	25\160	17	18	13	13	17	18	13	13
MIZ	mug100_25	100\166	5	18	5	5	6	13	5	4
MIZ	mug100_1	100\166	5	18	5	5	6	13	5	4
CAR	2-FullIns_3	52\201	16	20	12	16	16	18	12	8
MYC	r125.1	125\209	9	20	7	7	10	11	7	6
CAR	1-Insert._4	67\232	23	22	9	9	16	19	9	7
MYC	myciel5	47\236	24	22	13	13	22	21	13	9
SGB	jean	80\254	37	23	12	14	19	20	12	11
SGB	queen6_6	36\290	20	24	16	16	20	24	16	16
SGB	huck	74\301	54	25	11	21	28	22	11	11
CAR	3-FullIns_3	80\346	20	26	14	20	20	24	14	10
SGB	miles250	128\387	17	28	13	15	16	23	13	10
SGB	david	87\406	83	29	16	31	42	26	16	12
SGB	queen7_7	49\476	25	31	21	19	25	30	21	19
SGB	anna	138\493	72	31	15	51	37	28	15	12
CAR	4-FullIns_3	114\541	24	33	16	24	24	30	16	12
CAR	2-Insert._4	149\541	38	33	9	11	20	29	9	9
CAR	1-FullIns_4	93\593	33	34	18	33	26	33	18	13
SGB	games120	120\638	14	36	13	14	15	33	13	11
SGB	queen8_8	64\728	28	38	24	22	28	37	24	22
DSJ	dsjc125.1	125\736	24	38	17	20	24	36	17	12
MYC	myciel6	95\755	48	39	21	25	44	37	21	14
CAR	5-FullIns_3	154\792	28	40	18	28	29	37	18	14
MYC	r250.1	250\867	14	42	13	13	15	36	13	10
CAR	3-Insert._4	281\1046	57	46	9	13	29	40	9	9
SGB	queen9_9	81\1056	33	46	27	25	33	45	27	25
SGB	miles500	128\1170	39	48	29	35	35	47	29	25
CAR	1-Insert._5	202\1227	68	50	17	24	46	46	17	13
SGB	queen8_12	96\1368	33	52	31	30	33	51	31	27
SGB	queen10_10	100\1470	36	54	32	28	36	53	32	28
CAR	2-FullIns_4	212\1621	56	57	24	56	51	54	24	16
SGB	homer	561\1628	100	57	25	56	51	47	25	18
CAR	4-Insert._4	475\1795	80	60	9	15	41	52	9	9
SGB	queen11_11	121\1980	41	63	35	31	41	62	35	31
SGB	miles750	128\2113	65	65	42	55	57	64	42	37
MYC	myciel7	191\2360	96	69	35	49	88	67	35	23
SGB	queen12_12	144\2596	44	72	38	34	44	71	38	34
SGB	miles1000	128\3216	87	80	57	82	74	80	57	49
DSJ	dsjc250.1	250\3218	39	80	33	35	39	78	33	25
CAR	1-FullIns_5	282\3247	96	81	36	96	73	78	36	23
SGB	queen13_13	169\3328	49	82	43	37	49	81	43	37
CAR	3-FullIns_4	405\3524	85	84	28	85	72	80	28	20
REG	zeroin_i3	206\3540	141	84	41	38	119	83	41	32
REG	zeroin_i2	211\3541	141	84	41	38	119	83	41	32
DSJ	dsjr500.1	500\3555	26	84	23	26	27	79	23	18
MYC	r125.5	125\3838	100	88	61	70	85	87	61	52
REG	mulsol_i2	188\3885	157	88	53	34	139	87	53	33
DSJ	dsjc125.5	125\3891	76	88	63	72	72	88	63	57
REG	mulsol_i3	184\3916	158	89	54	34	140	87	54	33
REG	mulsol_i1	197\3925	122	89	65	82	111	87	65	51
CAR	2-Insert._5	597\3936	150	89	20	39	76	83	20	17
REG	mulsol_i4	185\3946	159	89	54	34	140	88	54	33
REG	mulsol_i5	186\3973	160	89	55	34	141	88	55	33
REG	zeroin_i1	211\4100	112	91	54	95	104	89	54	51
HOS	ash331GPIA	662\4185	24	91	20	23	25	85	20	16
SGB	queen14_14	196\4186	52	92	46	40	52	90	46	40
SGB	queen15_15	225\5180	57	102	49	43	57	101	49	43

Table 1 (*continued*)

Instances			Known upper bounds					New upper bounds		
Sour.	Name	$n\backslash m$	d	l	M	s	q	ξ	ζ	η
SGB	miles1500	128\5198	107	102	84	106	96	102	84	78
LEI	le450_5a	450\5714	43	107	34	35	44	104	34	25
LEI	le450_5b	450\5734	43	107	34	35	43	104	34	26
SGB	queen16_16	256\6320	60	112	54	46	60	111	54	46
CAR	1-Insert._6	607\6337	203	113	33	69	136	108	33	25
CAR	4-FullIns_4	690\6650	120	115	36	120	104	110	36	24
DSJ	dsjc125.9	125\6961	121	118	109	113	116	118	109	106
HOS	will199GPIA	701\7065	42	119	35	35	42	114	35	28
MYC	r125.1c	125\7501	125	122	116	116	123	122	116	116
HOS	ash608GPIA	1216\7844	21	125	20	20	22	116	20	16
LEI	le450_15a	450\8168	100	128	57	68	93	125	57	39
LEI	le450_15b	450\8169	95	128	56	72	88	125	56	39
LEI	le450_25a	450\8260	129	129	63	85	114	126	63	46
LEI	le450_25b	450\8263	112	129	60	80	99	126	60	43
REG	fpsol2i3	425\8688	347	132	53	68	299	130	53	35
REG	fpsol2i2	451\8691	347	132	53	68	299	129	53	35
CAR	3-Insert._5	1406\9695	282	139	25	58	142	130	25	17
LEI	le450_5d	450\9757	69	140	52	53	68	137	52	41
LEI	le450_5c	450\9803	67	140	52	55	67	138	52	41
CAR	5-FullIns_4	*1085\11 395*	161	151	49	161	142	145	49	28
REG	fpsol2i1	496\11 654	253	153	79	102	231	150	79	67
CAR	2-FullIns_5	852\12 201	216	156	56	216	193	152	56	31
DSJ	dsjc500.1	500\12 458	69	158	59	61	69	156	59	47
HOS	ash958GPIA	*1916\12 506*	25	158	21	22	26	147	21	17
REG	inithx_i3	621\13 969	543	167	52	235	476	164	52	38
REG	inithx_i2	645\13 979	542	167	52	235	476	164	52	38
MYC	r1000.1	*1000\14 378*	50	170	41	47	51	165	41	34
SCH	school1_nsh	352\14 612	233	171	101	115	195	170	101	84
MYC	r250.5	250\14 849	192	172	119	154	166	172	119	99
DSJ	dsjc250.5	250\15 668	148	177	126	134	141	177	126	116
LEI	le450_15c	450\16 680	140	183	93	129	133	181	93	70
LEI	le450_15d	450\16 750	139	183	92	129	131	182	92	70
LEI	le450_25c	450\17 343	180	186	101	128	163	185	101	76
LEI	le450_25d	450\17 425	158	187	99	138	145	185	99	75
REG	inithx_i1	864\18 707	503	193	074	239	441	190	74	57
SCH	school1	385\19 095	283	195	117	172	213	194	117	98
CUL	flat300_20_0	300\21 375	161	207	144	148	155	206	144	135
CUL	flat300_26_0	300\21 633	159	208	146	152	154	208	146	136
CUL	flat300_28_0	300\21 695	163	208	146	157	158	208	146	136
GOM	qg.order30	900\26 100	59	228	59	59	60	226	59	59
DSJ	dsjc250.9	250\27 897	235	236	219	224	228	236	219	214
MYC	r250.1c	250\30 227	250	246	238	242	246	246	238	236
CAR	3-FullIns_5	*2030\33 751*	410	260	79	410	343	253	79	40
KOS	wap05a	905\43 081	229	294	147	200	213	291	147	106
KOS	wap06a	947\43 571	231	295	147	200	211	293	147	105
DSJ	dsjc1000.1	*1000\49 629*	128	315	112	112	127	313	112	93
DSJ	dsjr500.5	500\58 862	389	343	234	282	347	343	234	197
GOM	qg.order40	*1600\62 400*	79	353	79	79	80	350	79	79
DSJ	dsjc500.5	*500\62 624*	287	354	251	260	277	353	251	236
HOS	abb313GPIA	*1557\65 390*	188	362	123	119	184	358	123	94
CAR	4-FullIns_5	*4146\77 305*	696	393	96	696	598	384	96	48
KOS	wap07a	*1809\103 368*	299	455	188	259	275	452	188	130
KOS	wap08a	*1870\104 176*	309	456	189	272	293	453	189	129
KOS	wap01a	*2368\110 871*	289	471	174	223	270	467	174	115
KOS	wap02a	*2464\111 742*	295	473	175	222	280	469	175	116
DSJ	dsjc500.9	*500\112 437*	472	474	443	450	461	474	443	437
DSJ	dsjr500.1c	*500\121 275*	498	492	478	489	490	492	478	476
GOM	qg.order60	*3600\212 400*	119	652	119	119	120	647	119	119
MYC	r1000.5	*1000\238 267*	782	690	472	535	696	690	472	396
CUL	flat1000_50	*1000\245 000*	521	700	492	503	511	700	492	474
CUL	flat1000_60	*1000\245 830*	525	701	493	501	515	701	493	472
CUL	flat1000_76	*1000\246 708*	533	702	494	501	523	702	494	474
DSJ	dsjc1000.5	*1000\249 826*	552	707	501	518	538	706	501	475
KOS	wap03a	*4730\286 722*	345	757	230	302	333	752	230	148
KOS	wap04a	*5231\294 902*	352	768	238	307	341	762	238	149

(*continued on next page*)

Table 1 (*continued*)

Instances			Known upper bounds					New upper bounds		
Sour.	Name	$n\backslash m$	d	l	M	s	q	ξ	ζ	η
LAT	latinsquare10	*900\307 350*	684	784	684	684	685	784	684	684
DSJ	dsjc1000.9	*1000\449 449*	925	948	888	912	910	948	888	877
MYC	r1000.1c	*1000\485 090*	992	985	957	976	978	985	957	951
GOM	qg.order100	*10 000\990 000*	199	1407	199	199	200	1401	199	199
MYC	c2000.5	*2000\999 836*	1075	1414	1000	1028	1054	1414	1000	962
MYC	c4000.5	*4000\4 000 268*	2124	2829	2002	2019	2093	2828	2002	1942
Average number of colors			186.1	218.5	122.2	147.5	171.2	215.9	122.2	108.9
Av. improvement of η (in%)			−46.1	−58.3	−18.4	−29.4	−42.8	−56.6	−18.4	0.0
Total time (s)			0.2	0.0	0.3	2.1	0.3	3.8	0.9	14.0

Table 2
Computational assessment of Propositions 1–6 based on Table 1.

Propositions		Avg. improvement (%)
Proposition 1	$\xi \leq l$	−4.56
Proposition 2	$\eta \leq \zeta$	−18.36
Proposition 3	$\zeta \leq d$	−35.99
Proposition 4	$\zeta = M$	0.00
Proposition 5	$\eta \leq s$	−29.39
Proposition 6	$\zeta \leq q$	−32.02

Proposition 6. *For any simple undirected graph*

$$\zeta \leq q.$$

Proof. We prove by contradiction that $\zeta \leq q$ by using Proposition 4.

$$\zeta = M = \max_{i \in X} \min(d_i + 1, i).$$

We denote by A and B the two subsets of X: $A = \{1, \ldots, \zeta\}$ and $B = \{\zeta + 1, \ldots, n\}$.
As shown in the proof of Proposition 4:

$$i \leq \zeta \leq d_i + 1 \quad \forall i \in A$$
$$d_i + 1 \leq \zeta \leq i \quad \forall i \in B.$$

We assume that $\zeta > q$.
First, it is recalled that Stacho has proved in [12] that $d_q < q$, i.e. $d_q + 1 \leq q$. Then $\zeta > q$ does not hold if $q \in A$.
Second, if q belongs to B it must satisfy $\zeta \leq q$ which conflicts with the hypothesis $\zeta > q$.
Consequently, this proves that $\zeta \leq q$. □

4. Computational assessment of these bounds

The new bounds introduced in this paper are compared to the five bounds of the literature on the DIMACS instances [5] for graph coloring. The detailed results are shown in Table 1. The first three columns of this table provide the instance source at DIMACS, its name, the number of nodes and the number of edges. The next eight columns show the upper bound on the number of colors provided by the five bounds of the literature, and the three upper bounds introduced in this paper. The last three rows of Table 1 show the average value of each bound on the DIMACS instances, the penultimate row is the average improvement provided by η over all the other bounds (note that these figures are not computed on the average numbers of colors), and the last row is the total amount of CPU time (in seconds) required for computing each bound on an Intel x eon processor system at 2.67 GH z and 8 G bytes RAM. Algorithms have been implemented in C ++ and compiled with gcc 4.11 on a Linux System.

Table 2 is displayed to assess the practical strength of Propositions 1–6. As each proposition is of the form $a \leq b$ (except Proposition 4), the last column of Table 2 indicates by which amount bound a is better than bound b (the average improvement is defined as the average value of $(a - b)/b$ over all the instances, in percent). Naturally, this amount is 0% in the particular case of Proposition 4, as it is an equality. It can be seen that ξ does not provide a significant advantage over l in practice.

However, Propositions 2, 3, 5 and 6 are stronger as the improvement is larger than 18%. More specifically, the best bound proposed in this paper outperforms the best upper bound of the literature by more than 18% in average. Proving that $M = \zeta$ is important for highlighting the reason for the practical superiority of η over M. Indeed, η is based on the same principle as ζ; it focuses on the degrees of saturation of nodes. The difference is that, η goes one step further than ζ by considering the degree of saturation of the neighbors of each nodes (i.e. the so-called neighboring density). This additional requirement has a computational cost which is drastically larger than the one required by computing ζ, but it provides a significant improvement in terms of the upper bound quality.

5. Conclusion

This paper introduces three new upper bounds on the chromatic number, without making any assumptions on the graph structure. The first one, ξ, is based on the number of edges and nodes and only requires connectivity, whereas ζ and η are based on the degree of the nodes in the graph. It is shown that ζ is equal to an existing bound, while being computed in a very different way. Moreover, a series of inequalities are proved, showing that these new bounds outperform five of the most well-known upper bounds from the literature. Computational experiments also show that the best bound proposed in this paper, η, is significantly better than the five bounds of the literature, and highlight the benefit of using the degree of saturation and its refined version (the neighboring density) for producing competitive upper bounds for graph coloring.

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